Fourier Analysis

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Review

Thm (Weyl).

Let γ be an irrational number. Then the sequence

$$\left(\{n\gamma\}\right)_{n=1}^{\infty}$$

is equidistributed in $[0,1)$.

Recall that a sequence $(x_n)_{n=1}^{\infty} \subset [0,1)$ is said to be equidistributed in $[0,1)$ if

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \le n \le N : \quad x_n \in (a,b)\} = b-a.$$

for all $(a,b) \subset [0,1)$.

More generally, Weyl proved the following result:

Thm (Weyl's criterion) Let $(x_n)_{n=1}^{\infty}$ be a sequence with $x_n \in [0,1)$. Then $(x_n)_{n=1}^{\infty}$ is equidistributed in $[0,1)$ if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k \cdot x_n} = 0 \text{ for all } k \in \mathbb{Z} \setminus \{0\}.$$

§ 4.4 A continuous but nowhere differentiable function on $\mathbb{R}$.

In 1861, Riemann conjectured that

$$R(x) = \sum_{n=0}^{\infty} \frac{\sin(2^n x)}{2^n}$$

is cts but nowhere differentiable on $\mathbb{R}$.

In 1916, Hardy proved that

$R(x)$ is not diff if $\frac{x}{\pi}$ is irrational.

In 1969, Gerver proved that

$R(x)$ is diff $\iff \frac{x}{\pi} = \frac{p}{q}$ where p, q are odd integers

In 1872, Weierstrass constructed the first example of cts but nowhere differentiable function:

$$f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n x),$$

where $a > 1$ is an integer, $0 < b < 1$ such that $ab > 1 + \frac{3\pi}{2}$.

Remark: Nowadays it is known that the conditions can be weaken to: $a > 1$, $0 < b < 1$ with $ab > 1$.

Here we proved a special version of Weierstrass' result.

Thm 1. Let $0 < \alpha < 1$. Define

$$f_\alpha(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} \cdot e^{i 2^n x}, \quad x \in \mathbb{R}.$$

Then f_α is cts but nowhere diff on $\mathbb{R}$.

(notice that $\widehat{f_\alpha}(m) \neq 0 \iff m = 2^n$ for some integer $n \geq 0$)

Idea: Let $g \in \mathcal{R}([-\pi,\pi])$ (i.e. g is integrable on $[-\pi,\pi]$).

We consider the so-called delayed mean of g,

$$\Delta_N(g)(x) = 2\cdot \sigma_{2N}(g)(x) - \sigma_N(g)(x),$$

where

$$\sigma_N(g)(x) = \sum_{n=-N}^{N} \left(1-\frac{|n|}{N}\right)\hat{g}(n)\, e^{inx}$$

(N-th Cesaro Mean of g)

By a direct calculation,

$$\begin{aligned}
\Delta_N(g)(x) &= 2\sigma_{2N}(g)(x) - \sigma_N(g)(x) \\
&= 2\cdot \sum_{|n|\le 2N} \left(1-\frac{|n|}{2N}\right)\hat{g}(n)\, e^{inx} \\
&\quad - \sum_{|n|\le N} \left(1-\frac{|n|}{N}\right)\hat{g}(n)\, e^{inx} \\
&= \sum_{n=-N}^{N} \hat{g}(n) e^{inx} + 2\sum_{N<|n|<2N} \left(1-\frac{|n|}{2N}\right)\hat{g}(n)\, e^{inx}
\end{aligned}$$

Let us consider $\Delta_N(f_\alpha)(x)$

- Observe that if $N = 2^m$ then

$$\Delta_N(f_\alpha)(x) = S_N(f_\alpha)(x)$$

$$= \sum_{n=0}^{m} 2^{-n\alpha} e^{i2^n x}.$$

- Moreover if $N = 2^m$ then

$$\Delta_{2N}(f_\alpha)(x) - \Delta_N(f_\alpha)(x)$$

$$= \sum_{n=0}^{m+1} 2^{-n\alpha} e^{i2^n x} - \sum_{n=0}^{m} 2^{-n\alpha} e^{i2^n x}$$

$$= 2^{-(m+1)\alpha} e^{i2^{m+1}x}.$$

Prop 3. Let $g \in \mathcal{R}([-\pi, \pi])$. Suppose g is differentiable at x_0. Then

$$\left| \sigma_N(g)'(x_0) \right| \leq C \cdot \log N,$$

where C is a constant.

We first use Prop 3 to prove Thm 1

Proof of Thm 1: Suppose on the contrary that f_α is diff at one point x_0.

Then

$$\Delta_N(f_\alpha)'(x_0) = 2\sigma_{2N}(f_\alpha)'(x_0) - \sigma_N(f_\alpha)'(x_0)$$

$$\begin{aligned} \left|\Delta_N(f_\alpha)'(x_0)\right| &\leq 2\left|\sigma_{2N}(f_\alpha)'(x_0)\right| + \left|\sigma_N(f_\alpha)'(x_0)\right| \\ &\leq 2 \cdot C \cdot \log(2N) + C \cdot \log N \\ &\leq \tilde{C} \log N. \end{aligned}$$

$*$

Taking $N = 2^m$,

$$\Delta_{2N}(f_\alpha)(x) - \Delta_N f_\alpha(x) = 2^{-(m+1)\alpha} \cdot e^{i 2^{m+1} x}$$

Then

$$\Delta_{2N}(f_\alpha)'(x_0) - \Delta_N f_\alpha'(x_0) = i\, 2^{(m+1)(1-\alpha)} \cdot e^{i 2^{m+1} x}$$

Hence

$$\left|\Delta_{2N}(f_\alpha)'(x_0) - \Delta_N f_\alpha'(x_0)\right| = 2^{(m+1)(1-\alpha)}. \quad (**)$$

However by $(*)$,

$$\left|\Delta_{2N}(f_\alpha)'(x_0)\right| \le \tilde{C} \cdot \log(2N) = \tilde{C} \cdot (m+1)$$

$$\left|\Delta_N(f_\alpha)'(x_0)\right| \le \tilde{C} \cdot \log(N) = \tilde{C} \cdot m$$

Hence

$$\left|\Delta_{2N}(f_\alpha)'(x_0) - \Delta_N f_\alpha'(x_0)\right| \le 2\tilde{C} m + \tilde{C}.$$

It leads to a contradiction with $(**)$. $\blacksquare$

In the end we prove Prop 3.

Lemma 4.

Let $$F_N(x) = \sum_{|n|\leq N} \left(1-\frac{|n|}{N}\right) e^{inx}$$

$$= \frac{\sin^2\left(\frac{N}{2}x\right)}{N \sin\left(\frac{x}{2}\right)^2}.$$

Then $\exists$ a constant $A>0$ such that

$$|F_N'(x)| \leq AN^2, \qquad |F_N'(x)| \leq \frac{A}{x^2} \qquad (***)$$

for all $x \in [-\pi, \pi]$.

Proof.

$$F_N(x) = \sum_{n=-N}^{N} \left(1-\frac{|n|}{N}\right) e^{inx}$$

$$F_N'(x) = \sum_{n=-N}^{N} \left(1-\frac{|n|}{N}\right) in\, e^{inx}$$

Hence $$|F_N'(x)| \leq \sum_{n=-N}^{N} \left(1-\frac{|n|}{N}\right)|n|$$

$$\leq (2N+1)N \leq 4N^2.$$

To see the other upper bound, notice

$$F_N(x) = \frac{\sin^2(\frac{N}{2}x)}{N\sin^2(\frac{x}{2})}.$$

So

$$F_N'(x) = \frac{\sin(\frac{N}{2}x)\cos(\frac{N}{2}x)}{\sin^2(\frac{x}{2})} - \frac{\sin^2(\frac{N}{2}x)\cdot\cos(\frac{x}{2})}{N\sin^3(\frac{x}{2})}.$$

Hence

$$|F_N'(x)| \leq \frac{1}{\sin^2(\frac{x}{2})} + \frac{|\sin(\frac{N}{2}x)|\cdot|\sin(\frac{N}{2}x)||\cos(\frac{x}{2})|}{N|\sin^3(\frac{x}{2})|}$$

$$\leq \frac{1}{\sin^2(\frac{x}{2})} + \frac{|\frac{N}{2}x|}{N|\sin^3(\frac{x}{2})|} \quad \left(\text{using } |\sin\alpha| \leq |\alpha|\right)$$

$$\leq A\cdot\frac{1}{x^2} \quad \left(\text{using } \frac{|\sin\alpha|}{|\alpha|} \geq \text{const} > 0 \text{ on } [-\tfrac{\pi}{2},\tfrac{\pi}{2}]\right)$$

Prop 3. Let $g \in \mathcal{R}([-\pi, \pi])$. Suppose g is differentiable at x_0. Then

$$\left| \sigma_N(g)'(x_0) \right| \leq C \cdot \log N,$$

where C is a constant.

Pf. Notice that

$$\sigma_N(g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(y) F_N(x-y)\, dy.$$

Taking derivative at x_0 gives

$$\sigma_N(g)'(x_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(y) \cdot F_N'(x_0-y)\, dy$$

(because F_N is smooth)

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x_0-y) F_N'(y)\, dy.$$

Notice that $\frac{1}{2\pi} \int_{-\pi}^{\pi} F_N'(y)\, dy = 0$ (since F_N is 2π periodic).

As a consequence

$$\sigma_N(g)'(x_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(g(x_0-y) - g(x_0)\right) F_N'(y)\, dy$$

Then

$$\left| \sigma_N(g)'(x_0) \right| \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| g(x_0-y) - g(x_0) \right| \left| F_N'(y) \right| dy$$

$$\leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} C \cdot |y| \cdot \left| F_N'(y) \right| dy$$

(Since g is diff at x_0, so

$$\frac{g(x_0-y)-g(x_0)}{y} \quad \text{is unif bdd on } [-\pi, \pi] \Big)$$

Now

$$\int_{-\pi}^{\pi} |y| \left| F_N'(y) \right| dy$$

$$= \int_{|y|<\frac{1}{N}} + \int_{\frac{1}{N}<|y|<\pi} |y| \left| F_N'(y) \right| dy$$

But

$$\int_{|y|<\frac{1}{N}} |y| \left| F_N'(y) \right| dy \leqslant \int_{|y|\leqslant\frac{1}{N}} \frac{1}{N} \cdot AN^2 \cdot dy$$

$$= 2A.$$

$$\int_{\frac{1}{N}<|y|<\pi} |y|\,|F_N'(y)|\,dy \leq \int_{\frac{1}{N}<|y|<\pi} |y| \cdot \frac{A}{|y|^2}\,dy$$

$$= \int_{\frac{1}{N}<|y|<\pi} \frac{A}{|y|}\,dy$$

$$= 2A \cdot \log|y| \Big|_{\frac{1}{N}}^{\pi}$$

$$= 2A\left(\log N + \log \pi\right)$$

Hence

$$\int_{-\pi}^{\pi} |y|\,|F_N'(y)|\,dy \leq \tilde{A}\left(\log N\right).$$

So

$$\left|\sigma_N(g)'(x_0)\right| \leq \tilde{\tilde{A}} \log N. \qquad \blacksquare$$